Bootstrap resampling for modeling financial risk

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This supplement summarizes a topic not discussed in the course packet: bootstrap resampling for modeling financial assets.

A multiple-asset problem. Recall that the idea of Monte Carlo simulation is to approximate complicated probability distributions via computer simulations. Up to now we've mainly focused on probability distributions for one variable. But the Monte Carlo method works for joint distributions of more than one variable, too. The example we'll consider here is where the variables $X_1, \ldots, X_D$ are D correlated asset returns in a financial portfolio, with joint distribution $P(X_1, \ldots, X_D)$.

Suppose that we're trying to understand the consequences of some asset-allocation decision among these D assets. Let $X_{j,t}$ be the random variable denoting the return of asset j during time period t, and that you're investing over a horizon from $t = 1$ to $t = T$. We'll use the letter W_t to denote your total wealth at time period t. Thus W_0 is your initial wealth, and W_T is your final wealth.

Clearly W_T is an extremely complicated random variable that's a function of all the intermediate-stage asset returns. The easiest way to model this random variable is via Monte Carlo simulation. Suppose that you take your initial wealth W_0 and allocate it so that your holdings in the jth asset are $W_{0,j}$. We'll phrase this allocation in terms of a set of portfolio weights c_j, which are numbers between 0 and 1 that reflect the desired fraction of your wealth invested in each asset:

$$W_{0,j} = c_j \cdot W_0 \,.$$

For example, let's say there are two assets, stocks (X_1) and bonds (X_2), so that $D = 2$. Suppose you want to put 70% of your wealth in stocks and 30% in bonds. Then you'd have $c_1 = 0.7$ and $c_2 = 0.3$. Your portfolio weights are constrained to be nonnegative and to sum to 1: $c_j \geq 0$ and $\sum_j c_j = 1$.[1]

[1] In the real world, you can sell assets short, which corresponds to a negative portfolio weight. You can also leverage yourself by borrowing money to invest, which corresponds to a portfolio weight that exceeds 1. So in reality these constraints are relaxed. But while professional investment managers do these things routinely, you almost surely won't do either of them while investing your retirement portfolio, so we won't deal with this extra complication here.

To model the trajectory of your wealth over time using Monte Carlo simulation, repeat the following simulation many times.

(1) For $t = 1, \dots, T$:

a. Simulate $(X_{t,1}, X_{t,2}, \dots, X_{t,D})$ from the joint distribution of asset returns.

b. Update $W_{t,j}$, the value of your holdings in asset j at time t, using the simulated returns $X_{t,j}$ in the simple interest formula:

$$W_{t,j} = W_{t-1,j} \cdot (1 + X_{t,j})$$

c. Optionally, rebalance your portfolio to the target allocation. This has two substeps:

i. Calculate your total wealth at time t:

$$W_t = \sum_{j=1}^{D} W_{t,j} \,.$$

ii. Re-allocate your total wealth W_t to the assets in proportion to their target weights:

$$W_{t,j} = c_j \cdot W_t \,.$$

(2) After T time periods, calculate final wealth W_T by summing the final holdings in each asset:

$$W_T = \sum_{j=1}^{D} W_{T,j} \,.$$

At the end of many simulations, you will have a collection of Monte Carlo samples $W_T^{(i)}$ for your final wealth.

Sampling complicated joint distributions. In order to carry out this simulation, we have to be able to sample from complicated joint distributions. In the special case of two assets (e.g. stocks and bonds), we have already used a bivariate normal model in Step 1(a) of the above process.

However, this strategy can easily break down. In general, using parametric probability models (like the bivariate normal) to describe complicated joint distributions is fraught with difficulty. A joint distribution is typically very complicated mathematically. We might be oversimplifying a lot by assuming something like a bivariate normal distribution (or its generalization to $D > 2$, called the multivariate normal). Simply put, the model may not fit. For a high-dimensional joint distribution (large value of D), this is typically the rule rather than the exception. It's really hard to find parametric probability models that provide a good fit to high-dimensional joint distributions.

To sample from complicated joint distributions while avoiding the oversimplification of parametric models, a very practical technique is called *bootstrap resampling.* Suppose we have M past samples of the random variables of interest, stacked in a matrix or spreadsheet:

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1D} \\ X_{21} & X_{22} & \cdots & X_{2D} \\ \vdots & & & \\ X_{M1} & X_{M2} & \cdots & X_{MD} \end{pmatrix} \tag{1}$$

where X_{ij} is the ith sample of the jth variable. In our context, the ith row of this spreadsheet gives the returns/interest rates of D correlated assets on a single day (or month or year, depending on the time period of interest).

The key idea of bootstrap resampling is the following. We may not be able to describe what the joint distribution $P(X_1, \ldots, X_D)$ is, but *we do know that every row of this X matrix is a sample from this joint distribution.* Therefore, instead of sampling from some theoretical model for the joint distribution, we will sample from the sample—i.e. we will bootstrap the past data. Every time we need a new draw from the joint distribution $P(X_1, \ldots, X_D)$, in Step 1a of the above algorithm, we randomly sample (with replacement) a single row of X. This would entail a modification of the above algorithm:

(1) For $t = 1, \ldots, T$:

a. Take a sample from the *empirical* joint distribution $(X_1, X_2, \ldots, X_D)$, by resampling one set of past returns from data collected at the appropriate time scale (e.g. daily if the time period t is measured in days, yearly if the time period is years). This entails sampling one row of the big X matrix in Equation 1.

b. Update $W_{t,j}$, the value of your holdings in asset j at step t, as before, using the sampled returns from step 1.

c. Optionally, rebalance your portfolio to the target allocation as before.

(2) As before.